STABILITY OF NONLINEAR FLEXURE WAVES IN CYLINDRICAL SHELLS

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In the weakly linear formulation of the problem, the equations of a cylindrical shell [1] for the axisymmetric case reduce to equations in normalized variables, which are used to derive the nonlinear Schrödinger equation for flexure waves. This equation is used to examine the problem of the instability of modulation waves. The wave number region at which the stationary structure of the nonlinear waves can be destroyed due to the effects of decay instabilities is determined.

If the wavelength is assumed to be much greater than the shell thickness and if only a geometric nonlinearity is considered, the nonlinear equations of a cylindrical shell for the axisymmetric case have the form

$$c^{-2}\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}+\mu k_0\frac{\partial w}{\partial x}=\frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right)^2; \tag{1}$$

$$c^{-2}\frac{\partial^2 w}{\partial t^2}+k_0^2 w+h^2\frac{\partial^4 w}{\partial x^4}-\mu k_0\frac{\partial u}{\partial x}=\frac{1}{2}\mu k_0\left(\frac{\partial w}{\partial x}\right)^2+ \\ +\frac{\partial}{\partial x}\left[\frac{\partial w}{\partial x}\left(\frac{\partial u}{\partial x}-\mu k_0 w\right)+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^3\right]. \tag{2}$$

Here u(x, t) and w(x, t) are the longitudinal and radial displacements of the mid-thickness surface from the undeformed state; $c^{-2}=\rho(1-\mu^2)E^{-1}$; ρ is the density; μ is Poisson's ratio; E is Young's modulus; $k_0=R^{-1}$; $h^2=h_1^2/12$; and R is the radius and h_1 is the thickness of the shell.

Following [2], we introduce normalized variables by replacing $\partial/\partial t\to p$ and taking the Fourier transform

$$\begin{pmatrix}u_k\\ w_k\end{pmatrix}=\frac{1}{2\pi}\int\begin{pmatrix}u\\ w\end{pmatrix}\exp(-ikx)\,dx.$$

Then the system (1) and (2) takes the operational form

$$(p^2+c^2k^2)u_k+i\mu c^2k_0kw_k=c^2f_{1k}; \tag{3}$$

$$-i\mu c^2k_0ku_k+\left(p^2+c^2\varkappa_k^2\right)w_k=c^2f_{2k}, \tag{4}$$

where $f_{1,2k}$ is the Fourier image of the right side of Eqs. (1) and (2) respectively; and $\varkappa_k^2=k_0^2+h^2k^4$.

From the system (3) and (4), we obtain the expressions

$$u_k=\frac{\Delta_u(p)}{\Delta(p)},\quad w_k=\frac{\Delta_w(p)}{\Delta(p)}. \tag{5}$$

Here

$$\Delta_u=c^2\left[\left(p^2+c^2\varkappa_k^2\right)f_{1k}-i\mu c^2k_0kf_{2k}\right];$$
$$\Delta_w=c^2\left[i\mu c^2k_0kf_{1k}+(p^2+c^2k^2)f_{2k}\right];\quad \Delta=\left(p^2+\Omega_1^2(k)\right)\left(p^2+\Omega_2^2(k)\right),$$

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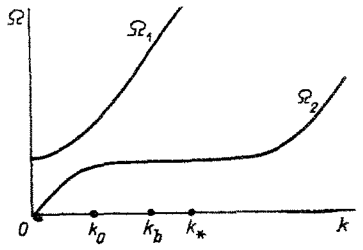

Fig. 1

where

$$\Omega_{1,2}^2(k) = \frac{c^2}{2}\left(k^2 + \varkappa_k^2 \pm r_k\right)\left(r_k = \sqrt{\left(k^2 - \varkappa_k^2\right)^2 + 4\mu^2 k_0^2 k^2}\right) \tag{6}$$

represents a dispersion relationship for the longitudinal and flexure waves respectively. The qualitative behavior of the dispersion curves $\Omega_{1,2}(k)$ is shown in Fig. 1.

The normalized variables are derived by expanding the right sides of Eqs. (5) into the simplest fractions:

$$u_k = \sum_s \left(d_k^s - i\beta_k b_k^s\right), \quad w_k = \sum_s \left(i\alpha_k d_k^s + b_k^s\right).$$

Here the index s takes the values + and −;

$$d_k^s = \frac{\Delta_u(-is\Omega_1)}{\Delta'(-is\Omega_1)(p + is\Omega_1)}; \tag{7}$$

$$b_k^s = \frac{\Delta_w(-is\Omega_2)}{\Delta'(-is\Omega_2)(p + is\Omega_2)}, \quad \Delta' \equiv \frac{d\Delta}{dp}. \tag{8}$$

The normalized variables d_k^s and b_k^s correspond to the longitudinal and flexure modes of oscillation. The coefficients

$$\alpha_k = \frac{\mu c^2 k_0 k}{c^2\varkappa_k^2 - \Omega_1^2(k)}, \quad \beta_k = \frac{c^2\varkappa_k^2 - \Omega_2^2(k)}{\mu c^2 k_0 k}$$

characterize the contribution of the longitudinal mode to the radial displacements and of the flexure mode to the longitudinal displacements respectively. We note that $\alpha_k \to 0$ and $\beta_k \to 0$ in the transition from a shell to a plate ($k_0 \to 0$).

According to (7) and (8), the normalized variables conform to the equations

$$\dot{d}_k^s + is\Omega_1(k)\, d_k^s = \frac{\Delta_u(-is\Omega_1)}{\Delta'(-is\Omega_1)}, \quad \dot{b}_k^s + is\Omega_2(k)\, b_k^s = \frac{\Delta_w(-is\Omega_2)}{\Delta'(-is\Omega_2)}.$$

Hereafter we only examine the process of nonlinear self-stimulation of flexure waves ($d_k^s \equiv 0$). If only cubic terms are considered on the righthand side, the equations for the flexure waves take the form

$$\dot{b}_k + i\omega_k b_k = i\int\{V_{kk_1k_2} b_{k_1} b_{k_2} d\lambda_{kk_1k_2} + 2V_{kk_1-k_2} b_{k_1}\bar{b}_{k_2} d\lambda_{kk_1-k_2} + \\ + V_{k-k_1-k_2}\bar{b}_{k_1}\bar{b}_{k_2} d\lambda_{k-k_1-k_2}\} + 3i\int T_{kk_1k_2-k_3} b_{k_1} b_{k_2}\bar{b}_{k_3} d\lambda_{kk_1k_2-k_3}, \tag{9}$$

where $b_k \equiv b_k^+$; $\omega_k \equiv \omega(k) \equiv \Omega_2(k)$; $d\lambda_{kk_1k_2} = \delta(k - k_1 - k_2)\, dk_1 dk_2$; $d\lambda_{kk_1k_2k_3} = \delta(k - k_1 - k_2 - k_3)\, dk_1 dk_2 dk_3$ and we consider the equation $b_{-k}^- = \bar{b}_k^+$, which follows from the condition that $w_{-k} = \bar{w}_k$ is real.

Calculations lead to the following expressions for the coefficients for nonlinear interaction:

$$V_{kk_1k_2} = \frac{1}{4\omega_k r_k}\{\mu c^2 k_0 k^2 k_1 k_2 + \left(c^2k^2 - \omega_k^2\right)[\mu k_0\left(k^2 - k_1k_2\right) - \\ - kk_1k_2\left(\beta_{k_1} + \beta_{k_2}\right)]\}, \quad T_{kk_1k_2k_3} = \frac{1}{4\omega_k r_k}\left(c^2k^2 - \omega_k^2\right) kk_1k_2k_3.$$

If we use the procedure [3], Eq. (9) for a spectrally narrow packet with the basic wave number q can lead to the nonlinear Schrödinger equation for the flexure wave packet $\psi(x, t)$ (in the coordinate system which moves with the group velocity):

$$i\frac{\partial\psi}{\partial t}+\frac{1}{2}\nu\frac{\partial^2\psi}{\partial x^2}+\sigma|\psi|^2\psi=0.$$

Here

$$\nu(q)=\frac{d^2\omega(q)}{dk^2};\quad \sigma(q)=4\frac{\omega_{2q}V_{q-q2q}V_{2qqq}}{\omega_{2q}^2-4\omega_q^2}+3T_{qqq-q}+\delta_q;\ \delta_q=(1-\mu^2)\times$$
$$\times\frac{v_0^2\left(c^2q^2-\omega_q^2\right)q^4}{2\omega_q r_q\left(v_0^2-v_q^2\right)};\quad v_0=\frac{d\omega(0)}{dk};\quad v_q=\frac{d\omega(q)}{dk}.$$

It is known [4] that the wave packet is unstable and that solitons of the flexure waves can form for $\nu\sigma > 0$. In the general case, the expressions for ν and σ are rather complex. However, in two particular cases, $q^2 \ll k_0^2$ (long waves) and $q^2 \gg k_0^2$ (short waves), it is possible to conduct an analytical investigation.

For long waves the packet is unstable, because

$$\omega_q=c\sqrt{1-\mu^2}\,q\left(1-\frac{1}{2}\mu^2q^2/k_0^2\right),\quad \nu(q)<0,\quad \sigma(q)=-\sqrt{1-\mu^2}\,cq^3/24<0.$$

If we consider the assumption that $h^2q^2 \ll 1$, which is necessary for the validity of Eqs. (1) and (2), then for short waves we have the relationships

$$\omega_q^2=c^2D_q^2,\quad D_q^2=(1-\mu^2)k_0^2+h^2q^4,\quad \nu(q)>0,$$
$$\sigma(q)=\frac{c^2q^2}{4\omega_q r_q}\left\{\frac{\mu^2k_0^2\left(D_{2q}^2-16D_q^2\right)^2}{16\left(D_{2q}^2-4D_q^2\right)}-(1+2\mu^2)q^4\right\}. \tag{10}$$

Here the coefficient $\sigma(q)$ is negative, except for a rather narrow range of (k_*, q_0) where $k_*^2 = 1/2\sqrt{1-\mu^2}k_0h^{-1}$, and $q_0 = k_*[1 + 4\mu^2k_0^2h^2/(1 + 2\mu^2)]$. If $q \to k_* - 0$, then $\sigma \to -\infty$; if $q \to k_* + 0$, then $\sigma \to +\infty$; and $\sigma(q_0) = 0$ at the point $q = q_0$. A small neighborhood ($|q - k_*| < a$, where $a < q_0 - k_*$) of the point k_* must be excluded from consideration, because the condition of weak nonlinearity of the problem formulation is invalid inside it. The singularity at the point $q = k_*$ results from a coincidence between the basic and doubling harmonics: $\omega(2k_*) - 2\omega(k_*) = 0$. It can be shown that the root of this equation, computed using the exact Eq. (6), lies namely in the short-wavelength region of wave numbers. Thus, the stationary structure of the short waves will be destroyed in the region $|q - k_*| < a$, due to the interaction with the secondary wave, and in the range $k_* + a < q < q_0$ due to the effects of the modulation instability.

Numerical calculation of the coefficients ν and σ for $R/h = 1600$ ($R/h_1 = 462$) and $\mu = 0.3$ confirms the results of the analytical investigation for long and short waves.

In order to study the stability in the intermediate range of wave numbers ($q \sim k_0$), we performed a numerical calculation of the coefficients ν and σ for the same values of R/h and μ. It was shown that $\sigma(q) < 0$ and $\nu(q) < 0$ for $q < k_b$, and $\nu(q) > 0$ for $q > k_b \approx 7k_0$ is the inflection point of the dispersion curve $\Omega_2(k)$ (see Fig. 1). The means that in this region the wave packet is unstable for $q < k_b$.

Investigation of the stability of the wave packet also requires considering the possibility of nonlinear three-wave interactions. For perturbations of a packet of flexure waves with a frequency $\omega(q)$ in a cylindrical shell, the packet will be unstable to interactions with a pair of waves [with frequencies $\omega_1(q_1)$ and $\omega_2(q_2)$, each of which can relate to both shear and longitudinal modes], if the following coincidence conditions

$$\omega(q)=\omega_1(q_1)+\omega_2(q_2); \tag{11}$$

$$q=q_1+q_2 \tag{12}$$

are fulfilled in correspondence with the dispersion relations (6). In this case a decay instability [5] takes place. From analysis of the dispersion curves (6), it follows that fulfillment of conditions (11) and (12) is possible for $q > 2k_*$. In particular if $k_0 < q_1 < k_*$ (let $q_1 \leq q_2$), it is possible to consider that $\omega_1(q_1) \approx \omega_0 = c^2(1 - \mu^2)k_0^2$ for the flexure branch $[\omega_1(q) \equiv \Omega_2(q)$ and also $\omega_2(q) \equiv \Omega_2(q)]$. Then by using (10), we find from (11) and (12) that $\omega_2 = \omega - \omega_0$, $q_2 = (\omega^2 - 2\,\omega\omega_0)^{1/4}\,(ch)^{-1/2}$, and $q_1 = q - q_2$. If we take $q < 2k_*$ for the same region of q_1 values and consider that $\omega(q) < \omega(2k_*) \approx 2\omega_0$, we obtain $\omega_1 + \omega_2 > \omega$. For values $q_1 < k_0$, such that $\omega_1(q_1) < \omega_0$, Eqs. (11) and (12) can be fulfilled only if $q \gg k_*$. The analogous situation arises in the case where the first longitudinal mode is selected for the first wave $[\omega_1(q) \equiv \Omega_1(q)]$.

Finally by summarizing the results obtained, we can conclude that a packet of flexure waves in a cylindrical shell is unstable in the following regions of wave numbers: $0 < q < k_b$, $k_* - a < q < q_0$, and $q > 2k_*$.

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ANALYSIS OF THRESHOLD-FREE FRACTURE OF MATERIALS ON REFLECTION OF A COMPRESSION PULSE FROM A FREE SURFACE

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Investigations of spalling phenomena yield information about the resistance of materials to fracture under microsecond loads. The most reliable and informative method is the method in which the velocity of a free surface is recorded continuously [1]. Figure 1 shows results of such experiments for plexiglass and rubber (curves 1 and 2) [2, 3]. The character of the spalling of plexiglass is typical for solids. After the shock wave reaches the free surface, the velocity profile repeats the shape of the compression pulse in the sample. When the tensile stresses reach a critical value, the material fractures, the stresses in the fracture zone decrease, and there appears a compression wave, which reaches the surface in the form of a spallation pulse. The subsequent velocity oscillations are due to the circulation of compression and rarefaction waves in the spalled plate. The fracture stress is determined by the difference between the maximum velocity of the surface and the velocity in front of the spallation pulse [1].

A fundamentally different result was obtained for rubber. According to Fig. 1, in this case the velocity decreases monotonically and characteristic oscillations are not observed. Since there is no clearly pronounced spallation pulse, there arises the question of how the fracture process should be characterized. If the strength of rubber were negligible, then after the shock wave reaches the free surface the velocity of the surface would remain constant. The dashed line in Fig. 1 shows the velocity profile, constructed assuming that the strength of rubber is high. The experimentally observed time dependence differs from the extreme cases by high and negligibly low dynamic tensile strength. The sample remaining after this experiment did not exhibit any clear indications of fracture.

It is known [4, 5] that rupture of elastomers is preceded by formation of microscopic nonuniformities in the sample, which starts at stresses much lower than the rupture stresses. The formation of nonuniformities is in itself still not fracture. Thus, in tests under tri-

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